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Inclusion Relations for Bernstein Quasi-analytic Classes

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The inclusion relations for classes of quasi-analytic functions introduced by S. N. Bernstein are studied. A criterion for determining when such a class is a subset of another is established. This is accomplished with the aid of a counting function associated with the class.

In an appendix to his famous book on the approximation of functions [1] S. N. Bernstein introduced certain new classes of quasi-analytic functions and gave a new proof of the Denjoy-Carleman theorem. The relations between these classes are studied in this article.

Let I be a closed interval of the real axis and $C(I)$ the Banach space of continuous real-valued functions on I with the usual norm:

$$\|f\| = \sup_{x \in I} |f(x)|.$$

For any function $f(x)$ in $C(I)$ let $E_n(f)$ be the distance from f to the subspace consisting of all polynomials of degree at most n. Evidently $\{E_n(f)\}$ is a monotone non-increasing sequence of non-negative numbers and the Weierstrass theorem guarantees that this sequence converges to 0. A remarkable result of S. N. Bernstein provides a converse result: if $\{a_n\}$ is a monotone non-increasing sequence of non-negative numbers converging to 0, then there exists a function $f(x)$ in $C(I)$ such that $E_n(f) = a_n$ for all n. (See, for example, [2, Sect. 2.5].)

The classes introduced by Bernstein are defined as follows. Let Λ be a

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monotone increasing infinite sequence of positive integers: then $C_\Lambda(I)$ consists of all continuous functions on I such that there exists a positive M and a number ρ in the interval (0, 1) so that

$$E_n(f) \leqslant M\rho^n \qquad \text{for all } n \text{ in } \Lambda.$$

It is easy to see that there is no loss of generality if we make the convention that every sequence Λ in question contains the integer 1. The family of all such sequences will be denoted by $\mathscr{S}$. For $\Lambda \in \mathscr{S}$, it is clear that $C_\Lambda(I)$ is a linear space. Bernstein [1, p. 163] established that membership in $C_\Lambda(I)$ is a local property, viz., that if a continuous function $f(x)$ belongs to both $C_\Lambda(I)$ and $C_\Lambda(J)$, where I and J are overlapping intervals, then f is a member of $C_\Lambda(I \cup J)$. However, we shall not need this fact in the sequel. The classes $C_\Lambda(I)$ are quasi-analytic in the sense that a function $f(x)$ in the class is uniquely determined by its values on any subinterval J of I. This is somewhat different from the usual Denjoy–Carleman classes of quasi-analytic functions since those classes consist of infinitely differentiable functions, while functions in $C_\Lambda(I)$ need not be infinitely differentiable, and in fact, as Bernstein remarks, may not be differentiable at all.

The class $C_\Lambda(I)$ is of course completely determined by the specification of the sequence $\Lambda \in \mathscr{S}$ although it is obvious that two different sequences may give rise to the same class. In this direction, Bernstein [1, p. 163] stated, without proof, the following result: Let Λ, $\Lambda^* \in \mathscr{S}$; then $C_\Lambda(I) = C_{\Lambda^*}(I)$ if and only if there exists a constant $M \geqslant 1$ such that for any $n \in \Lambda$ there exists an $n^* \in \Lambda^*$ so that

$$1/M \leqslant n^*/n \leqslant M.$$

As we shall see below, this result will follow as a corollary from our theorem.

We seek a criterion to determine when one such class is a subset of another. For this purpose it is convenient to introduce a "counting function" associated with a sequence $\Lambda \in \mathscr{S}$. This function is defined for $x \geqslant 1$ by the formula

$$\Lambda(x) = \sup[n \in \Lambda : n \leqslant x].$$

Evidently $\Lambda(x) \leqslant x$ with equality only when x is an integer in the sequence Λ. With this notation we have another, more convenient, way to state that a function $f(x)$ is a member of $C_\Lambda(I)$: this will be the case if and only if there exists a positive M and a number ρ in the interval (0, 1) such that $E_n(f) \leqslant M_\rho^{\Lambda(n)}$ for all positive integers n.

We can now state our result:

THEOREM. *Let Λ and Λ^* be two sequences in $\mathscr{S}$ and $\Lambda(x)$ and $\Lambda^*(x)$ the corresponding counting functions. The class $C_\Lambda(I)$ is a subset of $C_{\Lambda^*}(I)$ if and only if there exists a positive constant K so that $\Lambda^*(x) \leqslant K\Lambda(x)$ for all sufficiently large x.*

Proof. Suppose, first, that $f(x)$ is in $C_\Lambda(I)$ and that $\Lambda^*(x) \leqslant K\Lambda(x)$ for some suitable K and all sufficiently large x. Then for all sufficiently large integers n

$$E_n(f) \leqslant M\rho^{\Lambda(n)} \leqslant M\rho_1^{\Lambda^*(n)}, \qquad \text{where} \quad \rho_1 = \rho^{1/K}.$$

It follows that f is in $C_{\Lambda^*}(I)$, as desired.

On the other hand, if the class $C_\Lambda(I)$ is a subset of $C_{\Lambda^*}(I)$ we set $a_n = (1/2)^{\Lambda(n)}$ and invoke the theorem of Bernstein mentioned above to infer the existence of a continuous function $f(x)$ on I for which $E_n(f) = a_n = (1/2)^{\Lambda(n)}$ for all positive integers n. This function is evidently in $C_\Lambda(I)$, since we have only to take $M = 1$ and $\rho = \frac{1}{2}$ in the definition of that class. Accordingly, by hypothesis, f is in $C_{\Lambda^*}(I)$. This means that there exists a positive M and a number ρ in the interval $(0, 1)$ so that $E_n(f) = (1/2)^{\Lambda(n)} \leqslant M\rho^{\Lambda^*(n)}$ for all positive integers n.

If follows that

$$\Lambda(n)\log(1/2) \leqslant \log M + \Lambda^*(n)\log\rho$$

and therefore

$$\Lambda^*(n) \leqslant \Lambda(n)\frac{\log 2}{\log(1/\rho)} + \frac{\log M}{\log(1/\rho)}.$$

Since the function $\Lambda(x)$ is unbounded, we will have $\Lambda(x) \geqslant \log M/\log(1/\rho)$ for large enough x and therefore, for all sufficiently large positive integers,

$$\Lambda^*(n) \leqslant K\Lambda(n), \qquad \text{where} \quad K = \frac{\log 2}{\log(1/\rho)} + 1.$$

It is then easy to see that the same inequality holds for all sufficiently large x. This completes the proof.

Of course the inequality $\Lambda^*(x) \leqslant K\Lambda(x)$ for sufficiently large x is equivalent to the same inequality, perhaps with a larger K, for all $x \geqslant 1$. It is also clear that we have the following result.

COROLLARY. *Two classes $C_\Lambda(I)$ and $C_{\Lambda^*}(I)$ are the same if and only if there exists a constant $M \geqslant 1$ so that*

$$\frac{1}{M} \leqslant \frac{\Lambda^*(x)}{\Lambda(x)} \leqslant M \qquad \textit{for all } x \geqslant 1.$$

Note that this is precisely the aformentioned equivalence condition of Bernstein.

References

1. S. N. Bernstein, "Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réele," Gauthier–Villars, Paris, 1926.
2. A. F. Timan, "Theory of Approximation of Functions of a Real Variable," Pergamon, New York, 1963.